WAYS OF RANDOMIZING AND THE PROBLEM OF THEIR EQUIVALENCE

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ABSTRACT

Different ways of randomizing have been compared by various authors. An apparent discrepancy between the results stated by game theorists and those stated by statisticians is clarified here, and Kuhn's theorem on the necessity of perfect recall for the equivalence of two ways of randomizing is extended beyond countable cases.

1. Introduction

There are two different ways of introducing into a game the option of randomizing. In a game, given in its extensive form, we may permit a player to let his moves depend on extraneous random devices, or, normalizing the game first by introducing the set of (pure) strategies, the player may be permitted to choose an element of this set by employing a random device. It is only natural to enquire whether one of these two ways enables the player to achieve something not attainable through the other way.

This question has been studied successfully by a number of authors, and the results have passed into the realm of the well known. There is, however, a disturbing feature to the general knowledge: there is no agreement as to which direction of the question is trivial. This disagreement emanates from the original papers [1], [8], and some clarification seems in order. A portion of this paper is devoted to such a clarification, and it is followed by a new result, an extension of a converse theorem of Kuhn [6, Th. 4] to a more general setting.

For finite games, the problem of equivalence of the two ways of randomizing was settled by Kuhn [6], who introduced the concept of perfect recall, and proved

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its necessity and sufficiency for the equivalence of the two ways of randomizing. For games where perfect recall does not hold, randomizing the second way (that is, choosing a strategy at random) is more general.

Meanwhile, Wald and Wolfowitz [8] studied the problem in a setting more restricted in one sense and more general in another sense. Their result is established only for statistical games, and therefore the information structure has special simplifying features, but their information and action spaces include uncountable spaces. Subsequently, Kuhn's result on the sufficiency of perfect recall was generalized to the uncountable case by Aumann [1], who introduced appropriate definitions of randomization and of perfect recall for the uncountable setting. In terms of these definitions, randomization, while the game is in process, appears as a special case of choosing a strategy at random; the nontrivial part of the equivalence result consists of the fact that for perfect recall the greater generality of choosing a strategy at random is spurious. This is in striking contrast, not only to the terminology used by Wald and Wolfowitz, who follow von Neumann in calling randomization during the game *general* and randomization while choosing a strategy *special*, but also to the fact that in [8] the greater generality of general versus special is assumed to be self-evident, and the bulk of the paper is devoted to establishing the other direction of the equivalence, the one that becomes the trivial direction in terms of Aumann's definitions. Part of it is an attempt to untangle this discrepancy, by presenting Aumann's concepts and results, and comparing them with those of Wald and Wolfowitz. This is followed by a proof of the necessity of perfect recall in a setting that augments Aumann's by further measurability assumptions. Finally, convexity, the raison d'être of randomization, is brought into the picture. Some of the difficulties that appear when the moves of a game are permitted to occur at a set of times more general than the (well-ordered) positive integers, are faced in a forthcoming paper [4]; relations between the equivalence of ways of randomizing and the study of martingales and stopping times are also explored in [4].

2. Basic definitions

We retain most of Aumann's notation as well as the idea of a game normalized for all players but one, but we leave aside the payoff structure, which is not needed here. We use only one copy I of the unit interval, endowed with the Borel measurability structure, to serve in the role of information space and action space for all moves. Here a *game* consists, therefore, of an abstract set Z, the set of *strategies of the opponents*, and a sequence of *information functions* $g_1, g_2, \cdots$ where g_i is a function from $Z \times I^{i-1}$ into I, and is measurable for each fixed $z \in Z$. Intuitively, the player plays against a fixed z, unknown to him, and has to make his ith move by choosing $y_i \in I$ on the basis of knowing the numbers i and $g_i(z, y_1, \cdots, y_{i-1})$. Note that a fixed z represents not only the strategies that the other players have chosen, but also the strategy of chance as it acts in chance moves and in randomization devices that the other players may choose to use. In a statistical game, for example, z would determine not only the state of nature, but also the outcome of any sampling variable.

A game is of *perfect recall*, if for $i = 1, 2, \cdots$ there exist measurable *recall functions*, that is, functions u_i and t_i, from I into I, such that $u_i(g_{i+1}(z, y_1, \cdots, y_i)) = y_i$ and $t_i(g_{i+1}(z, y_1, \cdots, y_i)) = g_i(z, y_1, \cdots, y_{i-1})$ identically on $Z \times I^i$. (Compositions of subsequent u_i and t_i are named in [1], but it is not necessary to assume their existence separately.) Perfect recall is implied by the stronger property of perfect information, a property that cannot be defined in this framework of games normalized for the opponents of one player.

A (pure) *strategy* for a given game is a sequence m of measurable functions $m_1, m_2, \cdots$ from I into I. If the player uses m, and the opponents have chosen z, the player's moves $y_1, y_2, \cdots$ are defined inductively by $y_i = m_i(g_i(z, y_1, \cdots, y_{i-1}))$. Thus for given m, a mapping from Z to sequences $(y_1, y_2, \cdots)$ is defined; we call it the *effect* of m.

3. Random, behavior, and randomization strategies

Let Ω be I endowed with Lebesgue measure, and interpret it as the random device used by the player to choose a strategy at random. Alluding to the term random function we define a *random strategy* m as a sequence $m_1, m_2, \cdots$ of measurable functions from $\Omega \times I$ into I. It can be regarded as a pure strategy-valued random variable on Ω, or as a real-valued, measurable stochastic process, indexed by the pairs (i, g) with $g \in I$. The (finite) *joint distributions* of a random strategy are the joint distributions of the real-valued random variables $m_i(\cdot, g)$ for finitely many pairs (i, g) at a time. In addition to the usual compatibility relations between these joint distributions, further constraints are imposed on them by the assumed measurability of the stochastic process, that is, by requiring each m_i to be jointly measurable in $\omega \in \Omega$ and $g \in I$. There can never arise, for example, joint distributions under which every two $m_i(\cdot, g)$ are independent, or even uncorrelated [1]. There exist, however, random strategies whose $m_i(\cdot, g)$ are independent for pairs (i, g) with different i; more precisely: a random strategy is a *behavior strategy* if for any g_{ik}, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, k$ the random vectors

$$\begin{aligned} v_1 &= (m_1(\cdot, g_{11}), \cdots, m_1(\cdot, g_{1k})) \\ &\vdots \\ v_n &= (m_n(\cdot, g_{n1}), \cdots, m_n(\cdot, g_{nk})) \end{aligned}$$

are independent. (This is only assumed for two vectors with one component each in the original definition [1], but the results show that this stricter definition was intended; only with our corrected definition the following holds:) The behavior strategies are precisely those random strategies for which it is possible to replace Ω by a sequence of independent copies Ω_i, and to use a separate one for each move i, without altering the joint distributions of the strategy. Thus, while in general the randomization device Ω must be operated before the game starts, when a behavior strategy is played, part Ω_i of the device may be operated just before making the ith move. There is a different kind of strategy, which can only be used this way, randomizing for each move after the information available for making this move has been received. We call it a *randomization strategy*, and define it as a sequence $\mu_1, \mu_2, \cdots$ of measurable functions from I into the set of probability measures on I. The player uses it by choosing his ith move according to the distribution $\mu_i(g)$ if his information prior to this move was g, independently of anything else that may have occurred earlier.

4. Effective distributions and effective equivalence

For a pure strategy, its effect was defined as the rule that associated with each strategy z of the opponents, the sequence of moves y_i that the given pure strategy will lead the player to perform. Similarly, a random strategy m determines a *random effect*, which associates with each z the random sequence $y_1, y_2, \cdots$ defined by putting for each $\omega \in \Omega$ inductively

$$y_i(\omega) = m(\omega, g_i(z, y_1(\omega), \cdots, y_{i-1}(\omega))).$$

The *effective distribution* of m is the rule which associates which each z the distribution of the random sequence $y_1, y_2, \cdots$ that m and z determine. While the effective distribution of a random strategy is fully determined by the joint distributions of the strategy, the converse is usually false. Strategies with different joint distributions may be *effectively equivalent*, that is, their effective distributions may be the same.

Randomization strategies determine their effective distribution differently. For each z we first define the transition probabilities (to be regarded as the conditional distributions of y_i given $y_1, \cdots, y_{i-1}$) as the measures $\mu_i(g_i(z, y_1, \cdots, y_{i-1}))$. These transition probabilities uniquely determine for each z a distribution for the sequence $y_1, y_2, \cdots$ (see, for example, [7]).

The problem of equivalence of different ways of randomization can now be stated:

A. *For every given random strategy, is there an effectively equivalent randomization strategy?*

B. *For every given randomization strategy, is there an effectively equivalent random strategy?*

5. The results and their sources

The effective equivalence of random and randomization strategies was studied by Aumann [1] through the intermediary concept of behavior strategies. His version of Kuhn's theorem states that for a game of perfect recall, for every random strategy there is an effectively equivalent behavior strategy. As stated, the theorem asserts nothing about randomization strategies; it does assert a tautology for games where the player moves only once, since in this case every random strategy is a behavior strategy. Furthermore, since a behavior strategy is a special random strategy, there is no other direction to prove. However, the proof of the theorem contains more than its statement reveals. Paragraphs 6 and 8 essentially contain a proof of the fact that in a game of perfect recall, for every random strategy there is an effectively equivalent randomization strategy. Paragraph 7 essentially proves that in games where the player moves only once, there is for every randomization strategy an effectively equivalent random strategy. In paragraph 8 a sequence of such random strategies, one for each step of the game, are combined to form a behavior strategy. Joining these results, we obtain not only Aumann's version of Kuhn's theorem, but also an affirmative answer to (A) for games of perfect recall, and an affirmative answer to (B) for all games. This is made clear in Figure 1, where an arrow between boxes symbolizes the relation "for every element of the first box there exists an effectively equivalent element of the second box".

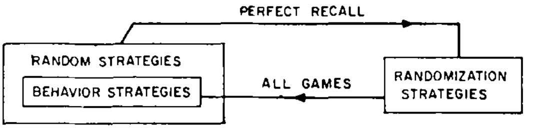

Fig. 1

Since Paragraph 7 uses the order structure of I, rather than just its measurability structure, its conclusion can be generalized to games with separable complete metric action spaces only by invoking the Borel-isomorphism of such spaces with I. Wald and Wolfowitz [8] do not use the isomorphism result when they prove such a generalization of the lower arrow of the diagram, and need therefore a more complicated construction than is used in Paragraph 7 by Aumann.

Since they deal with (sequential) statistical games, perfect recall is built into their definitions, and consequently the upper arrow holds. Its proof for this special case can be somewhat simplified, by making use of the fact that the only cases where the player must make a move in such games are the cases when all his past moves have been "continue", and so the infiniteness of the action space is never relevant for the past. Still, the claim that the upper arrow "follows at once from the definitions" [8] can hardly be supported and the application of von Neumann's terms ("special randomization" for random strategies, and "general randomization" for randomization strategies) to games with uncountable action and information spaces is only justified *ex post Aumann.*

Kuhn's original theorem [6] deals with the finite case, and states the appropriate special case of Aumann's result. His construction, however is different. In choosing the joint distribution for a random strategy with given effective distribution, Kuhn lets the $m_i(\cdot, g)$, for fixed i and different g, be independent. This is impossible in the uncountable context, and Aumann, Wald and Wolfowitz let all such $m_i(\cdot, g)$ be monotone functions of each other.

The ingenious construction in Blackwell and Girshick's book [2] establishes what can be done in fixed-sample-size statistical games when the player is restricted to using discrete randomization devices.

The necessity of perfect recall for the upper arrow was established by Kuhn [6] for the finite case. For the present framework, it is discussed below.

6. R-games and the necessity of perfect recall

Consider a game where $Z = I$, g_1 is nonmeasurable one-to-one I onto I, and g_2 is a Borel isomorphism of $I \times I$ onto I. The nonmeasurability of g_1 is permitted, since the g_i are assumed measurable for fixed z only. There exists a unique recall function t_1 for recalling g_1 namely, t_1 is the composition of g_1 with the first coordinate of the inverse of g_2. But this t_1 is nonmeasurable, and the game not of perfect recall. For this game there exists a random strategy, for which an effectly equivalent randomization strategy can be found only if we permit the latter to be nonmeasurable. For ω and g in I, define

$$m_1(\omega, g) = g + \omega (\text{mod } 1)$$

$$m_2(\omega, g) = 1 + u_1(g) - \omega (\text{mod } 1)$$

where u_1, the function for recalling y_1 is the (measurable) second coordinate of the inverse of g_2. For any z, the first move of the strategy is uniformly distributed on I, and the second move is $g_1(z)$ for all ω. The only effectively equivalent randomization strategy would be

$$\mu_1(g) = \text{Lebesgue measure on } I$$

$$\mu_2(g) = \text{measure 1 at the point } t_1(g)$$

but here $\mu_2(\cdot)$ is nonmeasurable.

This sort of pathology can be ruled out by adding further measurability assumptions, still without assuming a structure on Z. Consider the *recall relations*

$$U_i = \{(g_{i+1}, y_i) \mid (z, y_1, \cdots, y_i) \in Z \times I^i\} \quad \text{and}$$

$$T_i = \{(g_{i+1}, g_i) \mid (z, y_1, \cdots, y_i) \in Z \times I^i\}$$

which arise from the given information functions g_i. The game is of perfect recall if these relations are measurable functions, but we can separate the question of their measurability from their functionhood and assume measurability of the recall relations, whether they are functions or not. The recall relations are measurable if we assume that $g_{i+1}(Z \times B_i)$ is measurable for any Borel set $B_i \subset I^i$, and $g_{i+1}(g_i^{-1}(B) \times I)$ is measurable for any Borel set $B \subset I$. We call a game for which these assumptions hold an *R-game*, and prove for such games the following theorem.

THEOREM. *If in an R-game every random strategy has an effectively equivalent randomization strategy, the game is of perfect recall.*

PROOF. Recall functions can be found for an R-game, unless its information functions fail to "separate different past histories". This means that one of the following holds:

a. There are $z, y_1, \cdots, y_i$ and $z', y'_1, \cdots, y_i$ such that $g_{i+1}(z, y_1, \cdots, y_i) = g_{i+1}(z', y'_1, \cdots, y'_i)$ and for some $j \leqq i$, $y_j \neq y'_j$.

b. Measurable recall functions $u_1, u_2, \cdots$ exist, and there are $z, z', y_1, \cdots, y_i$ such that $g_{i+1}(z, y_1, \cdots, y_i) = g_{i+1}(z', y_1, \cdots, y_i)$ and $g_i(z, y_1, \cdots, y_{i-1}) \neq g_i(z', y_1, \cdots, y_{i-1})$.

For case a, let m be the pure strategy whose first i moves are $y_1, \cdots, y_i$, followed by the $(i + 1)$st move y_k, independently of the information available. Similarly, let m' have as its first $i + 1$ moves $y'_1, \cdots, y'_i, y'_j$. Let $\frac{1}{2}m + \frac{1}{2}m'$ be the random strategy that agrees with m when $\omega \leqq \frac{1}{2}$ and with m' when $\omega > \frac{1}{2}$. The effective distribution of this strategy does not depend on z, and its transition probabilities for the $(i + 1)$st move are the degenerate distributions at y_k and at y'_j, when the i first moves are given as $y_1, \cdots, y_i$ and $y'_1, \cdots, y'_i$ respectively. No randomization strategy can have such transition probabilities, since μ_{i+1}, being a function of g_{i+1}, will assign the same measure to $(z, y_1, \cdots, y_i)$ as to $(z', y'_1, \cdots, y'_i)$.

For case b, let both m and m' follow $y_1, \cdots, y_{i-1}$ as the first $i-1$ moves. Choose a number $t \neq y_i$, and let $m_i(g)$ be y_i when $g = g_i(z, y_1, \cdots, y_{i-1})$ and t when $g = g_i(z', y_1, \cdots, y_{i-1})$. Let $m'_i(g) = t + y_i - m_i(g)$. For the next move, put $m_{i+1}(g) = u_i(g)$, and $m_{i+1}(g) = t + y_i - u_i(g)$. Defining $\frac{1}{2}m + \frac{1}{2}m'$ as in case a, we obtain the following effective distribution: the first $i - 1$ moves are constant. The next two are (y_i, y_i) and (t, y_i) with probability $\frac{1}{2}$ each, for the opponents' strategy z, and (y_i, t) and (t, t) with probability $\frac{1}{2}$ each, for z. Thus, given $z, y_1, \cdots, y_i$, the next move is surely y_i, while given $z, y_1, \cdots, y_i$, it is surely t. Again, equality of g_{i+1} for these two sets of arguments precludes a randomization strategy having such effective distributions. Q.E.D.

7. Relation with convexity

For joint distributions of random strategies, there is a natural definition of convex combination. This definition makes the set of all joint distributions of random strategies a convex set, as can easily be seen by mixing two strategies, m_1 and m_2 as follows:

$$\text{let } m(\omega) = m_1(\omega/t), \text{ for } \omega \leqq t$$

$$\text{and } m(\omega) = m_1((\omega - t)/(1-t)), \text{ for } \omega > t$$

for constant $0 < t < 1$. The effective distribution depends on the joint distribution linearly, so the set K of effective distributions of all random strategies is also a convex set (in a different space). Now, compare it with the set $\boldsymbol{C}$ of effective distributions of all randomization strategies. By the lower arrow, $\boldsymbol{K} \supset \boldsymbol{C}$. By the upper arrow $\boldsymbol{C} \supset \boldsymbol{K}$ for games of perfect recall. By the proof of the converse theorem $\boldsymbol{C}$, is not convex when the game is an R-game not of perfect recall. Consequently, for R-games the following four statements are equivalent:

1. Perfect recall,
2. $\boldsymbol{K} = \boldsymbol{C}$,
3. C is convex,
4. $C \supset K$.

For all games, the pure strategies give rise to the extreme points of $\boldsymbol{K}$, and since pure strategies can also be regarded as special cases of randomization strategies, C always contains the extreme points of $\boldsymbol{K}$, and the implication from 3 to 2 can be related, after introducing appropriate topologies, to the Krein-Milman theorem [5], but we shall not pursue this relation here.

In a forthcoming paper we consider the implications of allowing the time index i to range over non-well-ordered sets. In particular an example of Dubins and Schwarz [3] set in the context of an appropriate definition of randomization strategies for negative integer time, gives rise to a game of perfect recall, with $\boldsymbol{K}$ and $\boldsymbol{C}$ both convex, and $\boldsymbol{K}$ a *proper* subset of $\boldsymbol{C}$, (one randomization strategy yields an extreme point of $\boldsymbol{C}$ which is not in $\boldsymbol{K}$) and the lower arrow fails to hold.

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